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ON SPACES OF FUNCTIONS BETWEEN CLASSIFYING SPACES

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ABSTRACT

In this paper the results of Dwyer and Zabrodsky [DZ] are extended by showing that if L is a compact Lie group and G is either a p-group or a torus, then every map $f : BG \to BL$ is homotopic to one induced by a homomorphism $\phi : G \to L$, and two such induced maps are homotopic if and only if the corresponding homomophisms are conjugate. Several other results related to maps between classifying spaces, completions, and fibrations are also deduced.

Introduction

In this paper we extend some of the results of [DZ], using a different approach. Our main theorem is the following:

THEOREM A: *Let L be a compact Lie group and let G be either a p-group or a torus. Then:*

(a) *Every map $f : BG \to BL$ is homotopic to a map $B\varphi : BG \to BL$ where $\varphi : G \to L$ is a homomorphism.*

(b) *Let $\varphi_0, \varphi_1 : G \to L$ be homomorphisms. Then $B\varphi_0 \sim B\varphi_1$ if and only if φ_0 and φ_1 are conjugate — that is, if and only if there exists an inner automorphism $a_g : L \to L$ such that $\varphi_1 = a_g \circ \varphi_0$.*

We shall use the following notations:

map (X, Y) - the space of (unpointed) maps from X to Y,
$\text{map}_*(X, Y)$ - the space of pointed maps from X to Y,
map $(X, Y)_f$ - the path component of $f : X \to Y$ in map (X, Y),

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$\mathrm{map}_*(X,Y)_f$ - the path component of $f:(X,*)\to(Y,*)$ in $\mathrm{map}_*(X,Y)$,
$\hat{C}(X,Y)=\mathrm{map}(X,Y)_0$ - the component of the constant map in $\mathrm{map}(X,Y)$,
$C(X,Y)=\mathrm{map}_*(X,Y)_0$ – the component of the constant map in $\mathrm{map}_*(X,Y)$,
$X_p^\wedge$ – the Bousfield–Kan completion of X with respect to the finite field $\mathbb{F}_p$,
$X^\wedge=\prod_p X_p^\wedge$, and similarly $\mathbb{Z}^\wedge=\prod_p\mathbb{Z}_p^\wedge$, where $\mathbb{Z}_p^\wedge=$ the p-adic integers,
$X_{\mathbb{Q}}$ – the rationalization of X.

We use the standard notations associated with a group G and G-spaces X, Y:

$$\begin{aligned} EG &- \text{ a free contractible } G\text{-space},\\ BG=EG/G &- \text{ the classifying space of } G,\\ X^G &- \text{ the fixed point set of } X,\\ \mathrm{map}_G(X,Y) &- \text{ the space of } G\text{-maps from } X \text{ to } Y,\\ X^{hG}=\mathrm{map}_G(EG,X) &- \text{ the homotopy fixed point set of } X. \end{aligned}$$

If L is a compact Lie group and $\varphi: G\to L$ is a homomorphism of groups, we have the following notations:

$$\begin{aligned} Z_L\varphi &- \text{ the centralizer of } \varphi G \text{ in } L,\\ N_L\varphi &- \text{ the normalizer of } \varphi G \text{ in } L,\\ W_L\varphi &= N_L\varphi/Z_L\varphi,\\ T_L &- \text{ a maximal torus in } L,\\ NT_L &- \text{ the normalizer of } T_L \text{ (in } L),\\ W_L &= NT_L/T_L - \text{ the Weyl group.} \end{aligned}$$

In the proof of Theorem A and its extensions we use the following theorems of Lannes, Miller, Carlsson, and Dwyer–Zabrodsky:

THEOREM 1 (Lannes [L, Thm. 7.1.1]): *Let X be a 1-connected space such that each $H^i(X,\mathbb{F}_p)$ is finite, and let V be an elementary abelian p-group. Then the natural map*

$$[BV,X]\to \mathrm{Hom}_{\mathcal{A}}(H^*(X,\mathbb{F}_p),H^*(BV,\mathbb{F}_p))$$

is bijective. (Here $\mathcal{A}$ denotes the category of unstable algebras over the mod-p Steenrod algebra.)

THEOREM 2 (Carlsson, Lannes, Miller; see, e.g., [M]): *Let G be a finite p-group acting simplicially on a finite simplicial complex X. Then the map $(X^G)^\wedge_p \to (X^\wedge_p)^{hG}$, induced by $X^G \to X^{hG}$, is a homotopy equivalence.*

THEOREM 3 (cf. [DZ], Thm 1.1): *Let G be a finite p-group, L a compact Lie group, $\varphi : G \to L$ any homomorphism, and consider the homomorphism $m : Z_L\varphi \times G \to L$ induced by φ, and the corresponding map $Bm : BZ_L\varphi \times BG \to BL$. Then the adjoint map $Bm_\# : BZ_L\varphi \to \mathrm{map}(BG, BL)_{B\varphi}$ induces a homotopy equivalence $(Bm_\#)^\wedge_p : (BZ_L\varphi)^\wedge_p \to \mathrm{map}(BG, (BL)^\wedge_p)_{B(\varphi^\wedge_p)}$.*

As a consequence of Theorem A and Theorem 3 one obtains:

THEOREM B: *Let $\varphi : T \to L$ be a homomorphism from a torus $T = T^n$ to a compact connected Lie group L. Then*
(a) *$\mathrm{map}_*(BT, BL)_{B\varphi}$ fibers principally over $(L/Z_L\varphi)^\wedge$ with a connected fiber homotopy equivalent to a finite product of Eilenberg–MacLane spaces whose homotopy groups are finitely generated $\mathbb{Z}^\wedge/\mathbb{Z}$ modules occurring in odd degrees. Consequently, the sufficiently high dimensional homotopy groups of $\mathrm{map}_*(BT, BL)_{B\varphi}$ are finite.*
(b) *The homotopy groups of the fiber of $\mathrm{map}(BT, BZ_L\varphi)_{B\varphi_0} \to \mathrm{map}(BT, BL)_{B\varphi}$ are rational vector spaces.*

THEOREM C: *Let $0 \to T \xrightarrow{\sigma} G \xrightarrow{\tau} W_G \to 1$ be a finite extension of a torus $T = T^n$, let L be a compact connected Lie group, and let $f : BG \to BL$ satisfy $f|BT \sim B\varphi$ for some homomorphism $\varphi : T \to L$. Then:*
(a) *There exists a homotopy commutative diagram*

$$\begin{array}{ccc} BT & \xrightarrow{B\varphi_0} & BZ_L\varphi \\ {\scriptstyle B\sigma}\downarrow & & \downarrow{\scriptstyle B\sigma'} \\ BG & \xrightarrow{f'} & BN_L\varphi \\ {\scriptstyle B\tau}\downarrow & & \downarrow{\scriptstyle B\tau'} \\ BW_G & \xrightarrow{B\alpha} & BW_L\varphi \end{array}$$

where $\alpha : W_G \to W_L\varphi$ is a homomorphism, $\varphi_0 : T \to Z_L\varphi$ is induced by φ, and f' covers f up to homotopy.
(b) *If $Z_L\varphi$ is a torus then $f' \sim B\varphi'$ for some homomorphism $\varphi' : G \to N_L\varphi$. In particular, if L is simple, any map $f : BL \to BL$ induces an endomorphism $\varphi_1 : NT_L \to NT_L$ such that $B\varphi_1$ is compatible with f.*

In the process of proving the main theorems, we prove some additional lemmas and propositions that may be interesting in their own right. For instance, the following theorem of Borel (see [B]) follows easily from Theorem 2:

COROLLARY 1.5: *Let G be a finite p-group of automorphisms of a compact connected Lie group L. Let $F \subset L$ be the subgroup of elements fixed by G and $F_0 \subset F$ the identity component. Then F/F_0 is a p-group. In particular, if G is a p-subgroup of L, then $Z_L G/(Z_L G)_0$ is a p-group.*

In the process of proving Theorem A, we prove the following:

PROPOSITION 1.3(a): *Let G and K be as in Theorem 2. If $\pi_1 K$ is finite, then $\pi_0 K^G \to \pi_0 K^{hG}$ is bijective.*

PROPOSITION 4.1: *Let $i : L_0 \subset L$ be a cocompact pair of topological groups, with $\pi_1(L/L_0)$ finite, and let G be a p-group.*
(a) *If $f : BG \to BL_0$ satisfies $Bi \circ f \sim B\varphi$ for some homomorphism $\varphi : G \to L$, then $f \sim B\varphi_0$, where $\varphi_0 : G \to L_0$ is a homomorphism and φ and $i \circ \varphi_0$ are conjugates.*
(b) *Let $\varphi_0, \varphi_1 : G \to L_0$ be homomorphisms. If $B\varphi_0 \sim B\varphi_1$ and $i \circ \varphi_0$, and $i \circ \varphi_1 : G \to L$ are conjugates, then φ_0 and φ_1 are, too.*

ORGANIZATION OF PAPER. Corollary 1.5 and Proposition 1.3 are proved in Section 1. Theorem 3 is proved in Section 3, using material from Section 1. Theorem A is proved in two parts: the case where G is a p-group is proved in Section 4, as is Proposition 4.1 (from which Theorem A(b) follows in this case). The case where G is a torus is proved in Section 6, using results from Section 2. $[BG, BL]$, for G a finite nilpotent group, is analyzed in Section 5; Section 7 contains the proof of Theorem B and Theorem C; and Section 8 contains a postscript by the editor.

Editor's note: It should be pointed out that a version of Theorem A was obtained independently by Dietrich Notbohm in his thesis (cf. [N]). ∎

1. Homotopy Fixed Points and Completions

In this section we study some simple conclusions of Theorem 2 relating the fixed points and homotopy fixed points of an (uncompleted) finite G-simplicial complex X. First notice the following:

LEMMA 1.1: *Let X be a G-CW complex. Then*

(a) *Every $h \in X^{hG}$, together with the inclusion $X^{hG} \subset map(EG, X) \simeq X$, induces an action of G on $\pi_n(map(EG, X), h) \cong \pi_n(X)$ and a homomorphism $\mathcal{U}_n : \pi_n(X^{hG}, h) \to \pi_n(X)^G$.*

(b) *For $n > 1$, every action of a group G on $K(A, n)$, where $K(A, n)^{hG} \neq \emptyset$, is equivalent to an action of G on the abelian (topological) group $K(A, n)$ by automorphisms. Hence $K(A, n)^{hG}$ is again an abelian group and*

$$\pi_m(K(A,n)^{hG}) \cong H^{n-m}(G, A) \quad \text{for } 0 \leq m \leq n \; .$$

For $n = 1$ one again has to assume $K(A,1)^{hG} \neq \emptyset$ and then $\pi_0 K(A,1)^{hG} \cong H^1(G, A)$, as sets, and $\pi_1(K(A,1)^{hG}, h) \cong A^G$ (where the action of G on A is determined by h).

(c) *If*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle v_0} \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

is a homotopy pullback diagram of G-spaces and G-maps, then so is

$$\begin{array}{ccc} X^{hG} & \xrightarrow{f^{hG}} & Y \\ {\scriptstyle v_1^{hG}}\downarrow & & \downarrow{\scriptstyle v_0^{hG}} \\ X_0^{hG} & \xrightarrow{f_0^{hG}} & Y_0^{hG} \end{array}$$

For $X_0 = EG$ one concludes: every $h \in Y_0^{hG}$ induces a G structure on a suitable representative X of the homotopy fiber of v_0, and X^{hG} is the homotopy fiber (over h) of v_0^{hG}. The G structure on X described above is characterized by the fact that $f : X \to Y$ is a G-map, and $v \circ f$ factors through EG.

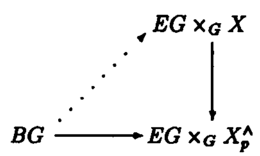
(d) *Let X be a nilpotent G-space and let $h \in X^{hG}$. If the G-module (G-group, for $n = 1$) $\pi_n X$ corresponding to h satisfies $H^m(G, \pi_n X) = 0$ for $n \geq m \geq 1$, then $\mathcal{U}_n$ of* (a) *is an isomorphism.*

(e) *Suppose $f : X \to Y$ is a G-equivariant map, and the homotopy fiber V of f is connected and has uniquely p-divisible homotopy groups. Suppose futher that $H^{k+i}(G, \pi_k V) = 0$ for $i = 0, 1$ and $k \geq 1$. Then $\pi_0 X^{hG} \cong \pi_0 Y^{hG}$, and the homotopy groups of the fiber are uniquely p-divisible.*

As a consequence of 1.1 one has:

PROPOSITION 1.2: (a) *Let X be a simply connected G-simplical complex. Then $\pi_0(X^{hG}) \to \pi_0((X_p^\wedge)^{hG})$ is bijective and the homotopy fibers of $X^{hG} \to (X_p^\wedge)^{hG}$ are connected and their homotopy groups are abelian and uniquely p-divisible.*
(b) *If X is nilpotent, then the set of path components of the homotopy fibers of $X^{hG} \to (X_p^\wedge)^{hG}$ is isomorphic to the set $(\pi_1 X_p^\wedge)^G/(\pi_1 X)^G = (\pi_1 X_p^\wedge/\pi_1 X)^G$, where $\pi_1 X_p^\wedge = (\pi_1 X)_p^\wedge \cong \pi_1(X_p^\wedge)$ is the p-completion of the nilpotent group $\pi_1 X$. The fundamental groups of these homotopy fibers are extensions of a finite group of order prime to p by an abelian uniquely p-divisible group.*

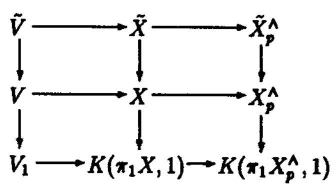
Proof: (a) If $V \to X \to X_p^\wedge$ is a fibration, then any G-map $EG \to X_p^\wedge$ has a unique lifting. To see this, note that the homotopy groups of the fiber V are (uniquely) p-divisible. Hence the lifting problem



has a unique solution, as the obstructions for existence lie in $H^{k+1}(G, \pi_k V)$, $k \geq 2$ and the obstructions for uniqueness lie in $H^k(G, \pi_k V)$, $k \geq 2$ – and all groups vanish by 1.1(b).

For the second assertion of 1.2(a), for any group H let $(\neq p)H$ denote the torsion subgroup of H modulo its p-torsion subgroup. Then one has exact sequences $0 \to (\pi_{n+1}X/\text{torsion}) \otimes \mathbb{Z}_p^\wedge/\mathbb{Z} \to \pi_n V \to (\neq p)\pi_n X \to 0$ and for every $h \in (X_p^\wedge)^{hG}$ the homotopy fiber over h in X^{hG} is V^{hG} by 1.1(c); now apply 1.1(e).

(b) One has a diagram



where $V_1 = K((\neq p)\pi_1(X), 1) \times \pi_1 X_p^\wedge/\pi_1 X$.

$\tilde{X}_p^\wedge$ is the universal covering space of $X_p^\wedge$. For $h \in (X_p^\wedge)^{hG}$ one obtains $h_1 \in K(\pi_1 X_p^\wedge, 1)^{hG}$ and one can see that $\pi_0 K(\pi_1 X, 1)^{hG} \xrightarrow{\cong} \pi_0 K(\pi_1 X_p^\wedge, 1)^{hG}$, hence one has $h_0 \in K(\pi_1 X, 1)^{hG}$ covering h_1. One has a fibration $\tilde{V}^{hG} \to V^{hG} \to V_1^{hG}$, where V^{hG} and V_1^{hG} are fibers over h and h_1 respectively, and $\tilde{X}$ is given a G-structure induced by h_0. (b) now follows from (a). ∎

PROPOSITION 1.3: *Let G be a p-group and X a finite G-simplical complex.*

(a) *Suppose either X is nilpotent or $\pi_1 X$ is finite: then $\pi_0 X^G \xrightarrow{\cong} \pi_0 X^{hG}$.*

(b) *If X^G, X are nilpotent and $\pi_1 X$, $\pi_1(X^G, x_0)$ are abelian for all $x_0 \in X^G$, then all the homotopy groups of the fiber of $X^G \to X^{hG}$ are abelian and uniquely p-divisible.*

Proof: (a) By Theorem 2, $\pi_0 X^G \xrightarrow{\cong} \pi_0(X_p^\wedge)^G \xrightarrow{\cong} \pi_0(X_p^\wedge)^{hG}$, hence $\pi_0 X^G \to \pi_0 X^{hG}$ is 1-1. For X 1-connected, $\pi_0 X^{hG} \to \pi_0(X_p^\wedge)^{hG}$ is a bijection by 1.2(a); thus $\pi_0 X^G \cong \pi_0 X^{hG}$. Otherwise, consider the universal covering space $\tilde{X}$: then $h \in X^{hG}$ induces a G-structure on $\tilde{X}$ and an element $\tilde{h} \in \tilde{X}^{hG}$. Hence, by the argument above, $\tilde{h} \sim_G \tilde{c}$ and $\tilde{c} \in \tilde{X}^G$. Hence $h \sim_G c$ and $c \in X^G$.
(b) The fibers V_1, V_2 of $X^G \to (X^G)_p^\wedge$ and $X^{hG} \to (X_p^\wedge)^{hG}$ respectively have the desired property, and the desired fiber is the same as that of $V_1 \to V_2$. ∎

PROPOSITION 1.4: *If X is a finite nilpotent G simplicial complex, then*

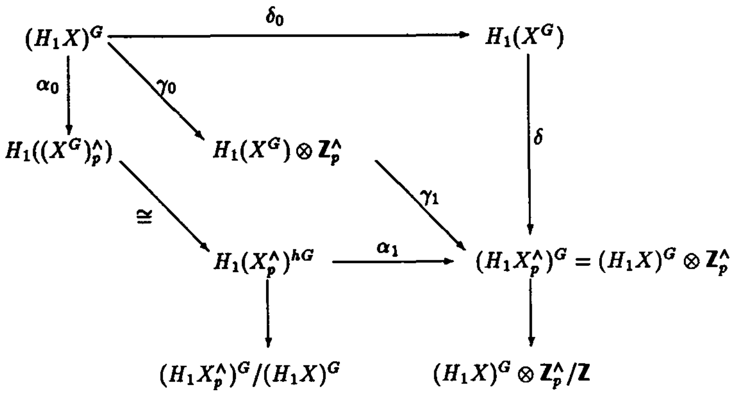
$$H_1(X^G) \to (H_1 X)^G$$

has a finite cokernel.

Proof: By 1.3 one has a surjection $\pi_0 X^{hG} \to \pi_0(X_p^\wedge)^{hG}$. This implies that $\pi_1(X_p^\wedge)^{hG} \to \pi_1((X_p^\wedge)^G)/\pi_1 X^G$ is a surjection, as is $H_1(X^G)_p^\wedge \xrightarrow{\cong} H_1(X_p^\wedge)^{hG} \to H_1(X_p^\wedge)^G/(H_1 X)^G$. Now $H_1((X^G)_p^\wedge) \to (H_1 X_p^\wedge)^G \subset H_1 X_p^\wedge$ factors through

$$\varprojlim H_1((\mathbb{F}_p)_s X^G) = H_1(X^G) \otimes \mathbb{Z}_p^\wedge$$

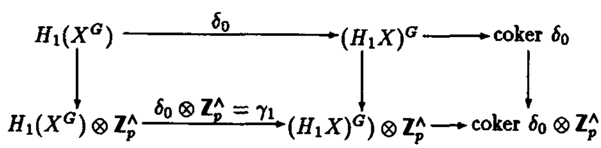
(where $\ldots (\mathbb{F}_p)_s X^G \to (\mathbb{F}_p)_{s-1} X^G \to \ldots$ is the tower of fibrations of Bousfield-Kan, whose inverse limit is $(X^G)_p^\wedge$). Thus one has



where im $\alpha_1 =$ imγ_1. $u =$ surjection implies

$$(H_1X_p^\wedge)^G = \mathrm{im}\alpha_1 + \mathrm{im}\delta = \mathrm{im}\gamma_1 + \mathrm{im}\delta.$$

We have



hence $\hat{\delta}$ is onto, which is equivalent to coker δ_0 being finite. ∎

COROLLARY 1.5: *Let G be a finite p-group of automorphisms of a compact connected Lie group L. Let $F \subset L$ be the subgroup of elements fixed by G and $F_0 \subset F$ the identity component. Then F/F_0 is a p-group. In particular, if G is a p-subgroup of L, then $Z_LG/(Z_LG)_0$ is a p-group.*

Proof: $F = L^G$ is a compact subgroup of L. $\pi_0 F = F/F_0$ is thus finite. $F \to \mathrm{map}_G(EG, L) = L^{hG}$, the map of constants, is a homomorphism; therefore, so is the function $\pi_0 F \xrightarrow{=} \pi_0 F_p^\wedge \to \pi_0(L_p^\wedge)^{hG}$, which by Theorem 2 is a bijection. Now $\pi_0(L_p^\wedge)^{hG}$ is a p-profinite group, while $\pi_0 F$ is finite. Hence both are finite p-groups. ∎

2. Representations and Homotopy Representation: The Classical Cases

The following special cases of groups satisfying the conclusion of Theorem A are given by the following classical facts:

PROPOSITION 2.1: *If G, H are finite groups or tori, then any map $BG \to BH$ is homotopic to a map of the form $B\varphi$.*

One can extend the classical case slightly:

PROPOSITION 2.2: *Let G be a finite group, and L a finite extension of a torus $T = T^n$ by a finite group W: $0 \to T \xrightarrow{\sigma} L \xrightarrow{\tau} W \to 1$. Then any map $f : BG \to BL$ is homotopic to $B\varphi$ for some homomorphism $\varphi : G \to L$.*

Proof: Let $0 \to V^n \xrightarrow{\sigma_0^n} T \xrightarrow{\times n} T \to 0$. Then V^n is characteristic, hence W acts on V^n, and if $|W|$ divides n then $H^2(W, V^n) \to H^2(W, T)$ is surjective. Hence one has a finite group L^n and a diagram:

$$\begin{array}{ccc} V^n & \xrightarrow{\sigma_0^n} & T \\ \downarrow & & \downarrow \\ L^n & \xrightarrow{\sigma^n} & L \\ \downarrow & & \downarrow \\ W & \xrightarrow{=} & W \end{array}$$

Now the obstruction to lifting a map

$$\begin{array}{ccc} & & BL^n \\ & \nearrow & \downarrow B\sigma^n \\ BG & \xrightarrow{f} & BL \end{array}$$

lies in $H^i(BG, \pi_{i-1}(\mathrm{Fiber}B\sigma^n))$. But $\mathrm{Fiber}B\sigma^n = \mathrm{Fiber}B\sigma_0^n = T$, and the only obstruction to the above lifting problem is $u_n \in H^2(BG, \pi_1 T = \mathbb{Z}^n)$. If $n|m$ one has

$$\begin{array}{ccc} & & BL^n \\ & \nearrow & \downarrow B\sigma^{n,m} \\ & & BL^m \\ & \nearrow & \downarrow B\sigma^m \\ BG & \longrightarrow & BL \end{array}$$

and $H^*(BG, B\sigma_0^{n,m})u_n = u_m$. Now the map $\mathrm{Fiber}B\sigma_0^n \to \mathrm{Fiber}B\sigma_0^m$ induces multiplication by m/n on $\pi_1 T = Z^n$, thus if $|G|$ divides m/n then $H^*(BG, B\sigma_0^{n,m})$

is trivial, so $u_m = 0$ and $f : BG \to BL$ lifts to $BG \to BL^m$. Since L^m is a finite group, one can apply 2.1. ∎

PROPOSITION 2.3: *Let L, L' be finite extensions of tori:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T^n & \longrightarrow & L & \longrightarrow & W & \longrightarrow & 1 \\ 0 & \longrightarrow & T^m & \longrightarrow & L' & \longrightarrow & W' & \longrightarrow & 1 \end{array}$$

Then any map $f : BL \to BL'$ is homotopic to $B\varphi, \varphi : L \to L'$.

To prove this proposition we first prove the following convenient Lemma:

LEMMA 2.4: *Let L be a topological group, W a finite group, and $1 \to W \xrightarrow{\sigma} \hat{L} \xrightarrow{\tau} L \to 1$ an exact sequence. Given a map $f : BL \to BL_0$, where L_0 is a compact Lie group, assume that $f \circ B\tau \sim B\hat{\varphi}$ for some homomorphism $\hat{\varphi} : \hat{L} \to L_0$; then $f \sim B\varphi$ for some $\varphi : L \to L_0$.*

Proof: One has a fibration $BW \to B\hat{L} \to BL$, and since by [M] the component $C_0(BW, BL_0)$ of the constant map in the function space $\mathrm{map}_*(BW, BL_0)$ is contractible, any map $\hat{f} : B\hat{L} \to BL_0$ with $\hat{f} \circ B\sigma \sim *$ factors uniquely (up to homotopy) through BL (see, e.g., [Z, 1.5]). If $\hat{f} = f \circ B\tau \sim B\hat{\varphi}$, then $\hat{\varphi} : \hat{L} \to L_0$ satisfies $B(\hat{\varphi} \circ \sigma) \sim *$.

But for finite groups W and compact Lie groups L_0 any $\alpha : W \to L_0$ is trivial if and only if $B\alpha \sim *$. (This is trivial for $L_0 = U(n)$ and $W = Z/nZ$, and the general case follows easily.) Hence $\hat{\varphi} \circ \sigma = *$ and $\hat{\varphi} = \varphi \circ \tau$, $\hat{L} \xrightarrow{\tau} L \xrightarrow{\varphi} L_0$. As $f \circ B\tau \sim B\varphi \circ B\tau$, by the above $f \sim B\varphi$. ∎

Proof of 2.3: Given $f : BL \to BL'$, one has a diagram

$$\begin{array}{ccc} BL & \xrightarrow{f} & BL' \\ \downarrow & & \downarrow \\ K(\pi_1 BL, 1) = BW & \xrightarrow{f'} & BW' = K(\pi_1 BL', 1) \end{array}$$

Then $f' \simeq B\varphi'$, and — replacing L' by the pullback

$$\begin{array}{ccc} L'_0 & \longrightarrow & L \\ {\scriptstyle \tau'_0}\downarrow & & \downarrow{\scriptstyle \tau'} \\ W & \xrightarrow{\varphi'} & W' \end{array}$$

if necessary — one may assume $W = W'$, $\varphi' = 1$. Now there exists a finite extension $\hat{W}$ of W, $\hat{W} \to W$ with lifting

$$\begin{array}{ccc} & & L \\ & \nearrow^{\cdots} & \downarrow{\scriptstyle \tau} \\ W & \longrightarrow & \hat{W} \end{array}$$

(e.g., the $\hat{W} = L^n$ of the proof of 2.2). For convenience, we thus may assume $\hat{W} = W$, $\hat{L} = L$ and one has $\chi : W \to L$. Now given $f : BL \to BL'$, one obtains a diagram

$$\begin{CD} BL @>f>> BL' \\ @VVV @VVV \\ K(\pi_1 BL, 1) = BW @>f_0>> BW' = K(\pi_1 BL', 1) \end{CD}$$

Again, $f_0 \sim B\varphi_0$ by Lemma 1.1. Hence, replacing L' by the pullback

$$\begin{CD} L' @>>> L \\ @VVV @VVV \\ W @>\varphi'>> W' \end{CD}$$

we see that f factors through $B\hat{L}'$, and again one may assume $\hat{L} = L'$ and one has

$$\begin{CD} BT^n @>>> BT^m \\ @VVV @VVV \\ BL @>f>> BL' \\ @VB\tau VV @VVB\tau'V \\ BW @>=>> BW \end{CD} \; .$$

Now, by Proposition 2.2 $f \circ B\chi : BW \to BL'$ is homotopic to $B\chi'$ for some $\chi' : W \to L'$, where $\tau' \circ \chi' = 1$. Since $BT^n \to BL$, $BT^m \to BL'$ are universal covering spaces, with W as group of covering transformations, one has a covering map $\tilde{f} : BT^n \to BT^m$ which is W-equivariant.

Now $\tilde{f} \sim B\varphi_2$ using the following procedure:

$$\begin{aligned} &[BT^n, BT^m] \xrightarrow{\cong} \mathrm{Hom}(H_2BT^n, H_2BT^m) = \mathrm{Hom}_C(T^n, T^m) \\ &\xrightarrow{B(-)} \mathrm{map}_*(BT^n, BT^m) \longrightarrow \pi_0\mathrm{map}(BT^n, BT^m) = [BT^n, BT^m] \, . \end{aligned}$$

Hence $\varphi_2 : T^n \to T^m$ is a W-equivariant homomorphism, $B\varphi_2 : BT^n \to BT^m$ is W-equivariant, BT^n has a W fixed point and φ_2 could be extended to a homomorphism $\varphi : L \cong T^n \rtimes W \to L' \cong T^m \rtimes W$ covering the identity on W.

Moreover, $BW \xrightarrow{B\chi} BL \xrightarrow{f} BL'$ is homotopic to $B\varphi \circ B\chi = B\chi'$ and $f|BT^n \sim B\varphi|BT^n$. Now $BL \cong EW \times_W BT^n$, hence $\mathrm{map}(BL, BL') \xrightarrow{\cong} [\mathrm{map}(BT^n, BL')]^{hW}$. The evaluation at the fixed point $ev : \mathrm{map}(BT^n, BL') \to BL'$ is W-equivariant (where BL' is a trivial W-space), and a homotopy equivalence of each component of $\mathrm{map}(BT^n, BL')$. Now, $f \circ B\sigma$ and $B\varphi \circ B\sigma = B\sigma \circ B\varphi_2$

are in the same path component in $\mathrm{map}(BT^n, BL')$ and $B\varphi \circ B\sigma$ being W-equivariant implies that this path component is W-invariant:

$$\mathrm{map}(BT^n, BL')_{B\varphi\circ\sigma} \xrightarrow{\cong} BL' .$$

$f, B\varphi \in [\mathrm{map}(BT^n, BL')_{B\varphi\circ\sigma}]^{hW} \xrightarrow{\cong} (BL')^{hW} = \mathrm{map}(BW, BL')$ have homotopic images $(f \circ B\chi, B\varphi \circ B\chi)$ in $(BL')^{hW}$ — hence are in the same path component in $\mathrm{map}(BL, BL') \cong \mathrm{map}(BT^n, BL')^{hW}$. ∎

3. Path Components of Function Spaces

If L is a topological group and G acts on L by automorphism, then

$$E(L \rtimes G) \to E(L \rtimes G)/L \simeq BL$$

is a G-map and principal L fiber bundle. This easily implies that for any free G simplicial complex K, $\mathrm{map}_G(K, E(L \rtimes G)) \to \mathrm{map}_G(K, E(L \rtimes G)/L = BL)$ is a Serre fibration, with fiber $\mathrm{map}_G(K, L)$. Taking $K = EG$ one obtains:

PROPOSITION 3.1: *Let L be a topological group and G a finite group acting on L by automorphisms. Then L^{hG} is a topological group and $B(L^{hG})$ is a path component of $(BL)^{hG}$. If $\sigma : L^G \to L^{hG}$ is the natural inclusion of groups, then $B\sigma : BL^G \to B(L^{hG}) \hookrightarrow (BL)^{hG}$ could be described as follows: $L^G \times G \to L \rtimes G$ induces G-maps $E(L^G) \times EG \to E(L \rtimes G)$ and $Bm : BL^G \times EG \to E(L \rtimes G)/L = BL$. $B\sigma$ is the adjoint of Bm.*

If G acts by inner automorphisms via $\varphi : G \to L$, then $L \rtimes G \cong L \times G$, $(BL)^{hG} \cong \mathrm{map}(BG, BL)$, $B(L^{hG}) \cong \mathrm{map}(BG, BL)_{B\varphi}$, and $B\sigma : B(L^G) = BZ_L\varphi \to B(L^{hG}) \cong \mathrm{map}(BG, BL)_{B\varphi}$ is the adjoint of $Bm : BZ_L\varphi \times BG \to BL$, $m : Z_L\varphi \times G \to L$ induced by φ.

Proof: $\mathrm{map}_G(EG, E(L \rtimes G)) = E(L \rtimes G)^{hG}$ is contractible; thus, its image in $(EL \rtimes G/L)^{hG}$ is a path component, which could be described as $B(L^{hG})$ — since here the fiber $\mathrm{map}_G(K, L) = L^{hG}$. If $\varphi : G \to L$ defines an action of G by inner automorphism, one has $L \times G \xrightarrow{\cong} L \rtimes G$ and a G-map $EG \to E(L \times G) = E(L) \times E(G)$ could be described as $E(\varphi, 1)$ for $(\varphi, 1) : G \to L \times G$. Then $BL^{hG} \cong \mathrm{map}_G(EG, (EL \times EG)/L) \cong \mathrm{map}_G(EG, BL \times EG)$ is easily seen to be equivalent to $\mathrm{map}(BG, BL)$, and $\mathrm{im}(\mathrm{map}_G(EG, EL) \to \mathrm{map}(BG, BL)) = \mathrm{map}(BG, BL)_{B\varphi}$. The identification of $B\sigma$ is clear. ∎

Putting Lemma 1.2(b) and Proposition 1.3 together with Proposition 3.1, one obtains:

THEOREM 3.2 (compare [DZ, Thm 1.1]): *Let G be a p-group acting on a compact connected Lie group L by automorphisms. Then the composite $B(L^G) \to B(L^{hG}) \hookrightarrow (BL)^{hG}$ induces an isomorphism on fundamental groups, and the homotopy groups of the fiber of this map are uniquely p-divisible.*

In particular, if $\varphi : G \to L$ describes an action by inner automorphisms, the above map could be identified with $Bm_\# : B(Z_L\varphi) \to \mathrm{map}(BG, BL)_{B\varphi}$. (Note that if $f : X \to Y$ induces an isomorphism in $\pi_1(-)$, then its homotopy fiber is nilpotent.)

THEOREM 3: *For G and L as above and $\varphi : G \to L$ any homomorphism, inducing $m : Z_L\varphi \times G \to L$ and $Bm : BZ_L\varphi \times BG \to BL$, the adjoint map $Bm_\# : BZ_L\varphi \to \mathrm{map}(BG, BL)_{B\varphi}$ induces a homotopy equivalence*

$$(Bm_\#)_p^\wedge : (BZ_L\varphi)_p^\wedge \to \mathrm{map}(BG, (BL)_p^\wedge)_{B(\varphi_p^\wedge)} .$$

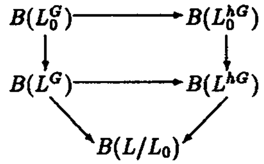
Proof: The homomorphism $\varphi : G \to L$ defines an action of G on L by conjugation, $g \cdot \ell = \ell^{\varphi(g)}$, and we can identify $Z_L\varphi$ with L^G. There is a homotopy equivalence $(L^G)_p^\wedge \to (L_p^\wedge)^G \xrightarrow{\sigma} (L_p^\wedge)^{hG}$ with the second map in the role of σ of 3.1. Theorem 3 follows by taking classifying spaces and making the identifications of 3.1. ∎

Remark 3.3: Theorem 3.2 is valid for nonconnected Lie groups L, too: if L_0 is the identity component of L, and $\rho : L \to L/L_0$ is the quotient map, let $L_1 = \rho^{-1}((L/L_0)^G)$. Then $(L_1)^G = L^G$ and $L_1^{hG} = L^{hG}$, and obviously $B(L_1^{hG}) = B(L^{hG}) \to (BL)^{hG}$ is a path component. Thus one can replace L by L_1, or equivalently assume that G acts trivially on L/L_0, and that each component L_α of L is a G-space.

If $L_\alpha^G = \emptyset$, also $L_\alpha^{hG} = \emptyset$ (see e.g. [DFZ, J]). Thus one can further reduce L_1, if necessary, to assume that $L^G \cap L_\alpha \neq \emptyset$ for each component L_α of L. Hence $L^G \to L/L_0$ is surjective and the square in

$$\begin{array}{ccc} L_0^G & \longrightarrow & L_0^{hG} \\ \downarrow & & \downarrow \\ L^G & \longrightarrow & L^{hG} \\ & \searrow \quad L/L_0 \quad \swarrow & \end{array}$$

is a pullback of groups, so the following is a pullback of spaces:



and 3.2 for L_0 implies 3.2 for L. ∎

PROPOSITION 3.4: *Let G be a p-group and V a subgroup of its center, with $i : V \subset G$ the inclusion. If $\varphi : V \to L$ is a homomorphism into a Lie group L, let $\mathrm{map}(BG, BL)_{(\varphi)}$ denote the set $\{f | f \circ Bi \sim B\varphi\}$. If $L_0 \subset L$ is a subgroup containing $Z_L\varphi$ — so φ factors through $\varphi_0 : Z_L\varphi \to L_0$ — then there is an isomorphism $\pi_0\mathrm{map}(BG, BL_0)_{(\varphi_0)} \xrightarrow{\simeq} \pi_0\mathrm{map}(BG, BL)_{(\varphi)}$.*

Proof: One has maps $BZ_L\varphi \xrightarrow{(Bm_0)_\#} \mathrm{map}(BV, BL_0)_{B\varphi_0} \xrightarrow{(Bi)_*} \mathrm{map}(BG, BL)_{B\varphi}$. As $(Bm_0)_\#$ and $(Bi)_* \circ (Bm_0)_\# = (Bm)_\#$ induce isomorphisms on $\pi_1(-)$ by Theorem 3.2, so does $(Bi)_*$; and the homotopy groups of the homotopy fiber of $(Bi)_*$ are uniquely p-divisible. Now G/V acts simply on BV — that is, for every $\hat{g} \in G/V$, the map $\hat{g}_*$ is homotopic to the identity. Hence one obtains an action of G/V on all function spaces $\mathrm{map}(BV, X)$, preserving path components. In particular, $(Bi)_*$ is G/V-equivariant. By Lemma 1.1(d), there is an isomorphism:

$$\pi_0(\mathrm{map}(BV, BL_0)^{h(G/V)}_{B\varphi_0}) \xrightarrow{\cong} \pi_0(\mathrm{map}(BV, BL)^{h(G/V)}_{B\varphi}) .$$

But we have $\mathrm{map}(BV, BL_0)^{h(G/V)}_{B\varphi_0} = \mathrm{map}(BG, BL_0)_{(\varphi_0)}$ and similarly

$$\mathrm{map}(BV, BL)^{h(G/V)}_{B\varphi} = \mathrm{map}(BG, BL)_{(\varphi)}.$$ ∎

PROPOSITION 3.5: *Let G be a finite p-group and $\varphi : G \to L$ a homomorphism into a compact Lie group L.*

(a) *Given a connected subgroup $L_0 \subset L$ with $Z_L\varphi \subset L_0$, such that $\varphi = i \circ \varphi_0$ for $\varphi_0 : G \to L_0$ and $i : L_0 \subset L$, one has a homotopy equivalence*

$$(Bi)_* : \mathrm{map}(BG, B(L_0{}^\wedge_p)_{B\hat{\varphi}_0} \xrightarrow{\simeq} \mathrm{map}(BG, B(L^\wedge_p)_{B\hat{\varphi}}$$

(b) *If G is abelian and $Z_L\varphi$ is connected, then there is a homotopy equivalence*

$$B((Z_L\varphi)^\wedge_p) \simeq \mathrm{map}(BG, B((Z_\varphi)^\wedge_p)_{B\hat{\varphi}_0} .$$

Proof: For (a) we have $Z_L\varphi \times G \to L$ factoring through L_0 and $Z_{L_0}\varphi_0 = Z_L\varphi$. Then (a) follows from Theorem 3. For (b), take $L_0 = Z_L\varphi$ and apply (a). ∎

4. Proof of Theorem A for G a p-Group

PROPOSITION 4.1: *Let $i : L_0 \hookrightarrow L$ be an inclusion of topological groups, and $\varphi : G \to L$ a homomorphism of a p-group into L. Assume:*

(i) *L/L_0 is a finite simplicial complex.*

(ii) *$\pi_1(L/L_0)$ is finite.*

(iii) *$L \to L/L_0$ has a local cross section.*

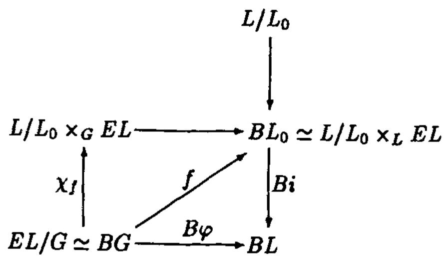
(iv) *The action of G on L/L_0 induced by φ is simplicial.*

Then:

(a) *If $f : BG \to BL_0$ is a homotopy lifting of $B\varphi$, then $f \sim B\varphi_0$ for some $\varphi_0 : G \to L_0$, and $i \circ \varphi_0$ is conjugate to φ.*

(b) *If $\varphi_0, \varphi_0^1 : G \to L_0$ satisfy $B\varphi_0 \sim B\varphi_0^1$ and $i \circ \varphi_0$, $i \circ \varphi_0^1$ are both conjugate to φ, then φ_0 and φ_0^1 are conjugates in L_0.*

Proof: (a) Consider:



where the section χ_f corresponds to an element $\tilde{\chi} \in (L/L_0)^{hG}$. By Proposition 1.3 $\tilde{\chi} \sim \tilde{\chi}_0$, where $\tilde{\chi}_0$ is a constant — say $EG \to \{xL_0\} \subset (L/L_0)^G$. Hence $x^{-1}(\varphi G)x \subset L_0$, and consequently $f \sim f_0$, where $f_0 : EL/G \simeq BG \to L/L_0 \times_L EL \simeq BL_0$ has the form $f_0([u]_G) = [xL_0, u]_L \in L/L_0 \times_L EL$ for $u \in EL$. The identification $L/L_0 \times_L EL \simeq EL_0/L_0 = BL_0$ is given by $[yL_0, u]_L \equiv [y^{-1}u]_{L_0}$; hence, $f_0[u]_G = [x^{-1}u]_{L_0}$ is covered by a map $EL \to EL$ defined by $u \mapsto x^{-1}u$, which is a φ_0-map. (Here $\varphi_0 : G \to L_0$ is given by $G \xrightarrow{\varphi} \varphi G \to x\varphi G x^{-1} \subset L_0$.) Clearly $i\varphi_0$ and φ are conjugates.

(b) Suppose $\varphi_0^1 = x^{-1}(i \circ \varphi_0)x$, for $x \in L$, so that $\varphi_0^1 G \subset L_0 \cap xL_0x^{-1}$. Let EL be considered a G-space via φ_0; then the identity $EL \to EL$ is a φ_0-map, and $R_{x^{-1}} : EL \to EL$ is a φ_0^1-map, where $R_{x^{-1}}(u) = x^{-1}u$. Now as in part (a), $B\varphi_0 \sim B\varphi_0^1$ implies that the two maps $\chi_0 : [u]_G \mapsto [L_0, u]_G$ and $\chi_0^1 : [u_G] \mapsto [x^u L_0, u]_G$ are homotopic.

The homotopy between them covers a G-homotopy between $\tilde{\chi}_0 : u \mapsto (L_0, u)$ and $\tilde{\chi}_0^1 : u \mapsto (\chi(u)L_0, y(u))$. But $[\chi(u)L_0, y(u)]_G = [xL_0, u]_G$, so $x(u)L_0 = gxL_0$ for some $g \in G$. Since $gxL_0 = xL_0$, it follows that the two constant G-maps $EL \to \{L_0\} \subset L/L_0$ and $EL \to \{xL_0\} \subset L/L_0$ are G-homotopic. Consequently L_0 and xL_0 are in the same path component in $(L/L_0)^G$, by Proposition 1.3.

Now $N_L\varphi_0$ acts on $(L/L_0)^G$ and $N_L\varphi_0 \setminus (L/L_0)^G$ may be identified with the set of L_0-conjugacy classes of the L-conjugates of $\varphi_0 G$. Following arguments of Quillen, this set may be also identified with a subset of the conjugacy classes of isotopy subgroups of the right L_0 action on $G \setminus L$. Since the quotient of this action is compact, this set is finite.

Moreover, since $|N_L\varphi_0 : Z_L\varphi_0| < \infty$, the set $Z_L\varphi_0 \setminus (L/L_0)^G$ is finite, and consequently xL_0 and L_0 are in the same class in $Z_L\varphi_0 \setminus (L/L_0)^G$.

Thus there is a $Z_0 \in Z_L\varphi_0$ such that $Z_0 x L_0 = L_0$ for $x = Z_0^{-1}x_0$ for some $x_0 \in L_0$. Thus $\varphi_0^1 = x^{-1}\varphi_0 x = x_0^{-1} Z_0 \varphi_0 Z_0^{-1} x_0 = x_0^{-1}\varphi_0 x_0$, so φ_0 and φ_0^1 are conjugate. ∎

COROLLARY 4.2: *Let $\varphi_1, \varphi_2 : G \to L$ be two homomorphisms of a p-group into a compact Lie group. If $B\varphi_1 \sim B\varphi_2$, then φ_1 and φ_2 are conjugates. (The converse is obvious.)*

Proof: Embed $i : L \subset U(n)$; then $Bi \circ B\varphi_1 \sim Bi \circ B\varphi_2$ implies that the representations $i \circ \varphi_i$, $i \circ \varphi_2 : G \to U(n)$ have the same image in $K^0(BG)$. By [A], the homomorphism $R(G) \to K_0(BG)$ is injective (G being a p-group); hence one may assume that $i \circ \varphi_1$ and $i \circ \varphi_2$ are conjugate. Now embed $U(n)$ in $SU(n+1)$: since $SU(n+1)/L$ has a finite fundamental group, Proposition 4.1 applies. ∎

LEMMA 4.3: *Theorem A(a) holds for $G = \mathbb{Z}/p$ and $L = SU(n)$ or $U(n)$.*

Proof: Using Lannes's Theorem 1, it suffices to show that any $\mathcal{A}$-morphism

$$\psi : H^*(BL, \mathbb{F}_p) \to H^*(B\mathbb{Z}/p, \mathbb{F}_p)$$

is of the form $\psi = H^*(B\alpha, \mathbb{F}_p)$ for some homomorphism $\alpha : \mathbb{Z}/p \to L$. (Assuming this, and given $f : B\mathbb{Z}/p \to BL$, one will have $H^*(f, \mathbb{F}_p) = H^*(B\varphi, \mathbb{F}_p)$ and the injectivity of $[B\mathbb{Z}/p, BL] \to \mathrm{Hom}_{\mathcal{A}}(H^*(BL, \mathbb{F}_p),\ H^*(B\mathbb{Z}/p, \mathbb{F}_p))$ will imply

$f \sim B\varphi$.) Following Adams and Wilkerson (cf. [AW]), any morphism

$$\begin{aligned}\psi : H^*(BL, \mathbf{F}_p) &= \mathbf{F}_p[x_{2m_1}, x_{2m_2}, \cdots, x_{2m_r}] \\ &\to H^{\text{even}}(B\mathbf{Z}/p, \mathbf{F}_p) \subset H^*(B\mathbf{Z}/p, \mathbf{F}_p)\end{aligned}$$

factors through $H^*(BT_L, \mathbf{F}_p)$ where $i : T_L \subset L$ is a maximal torus:

$$\begin{array}{ccc} H^*(BL, \mathbf{F}_p) & \xrightarrow{H^*(Bi)} & H^*(BT_L, \mathbf{F}_p) \\ & \searrow^{\psi} & \downarrow \\ & & H^*(B\mathbf{Z}/p, \mathbf{F}_p) \end{array}$$

But for a torus T, any morphism $\psi : H^*(BT, \mathbf{F}_p) \to H^*(B\mathbf{Z}/p, \mathbf{F}_p)$ is of the form $H^*(B\alpha, \mathbf{F}_p)$. ∎

4.4. *Proof of Theorem A(a) for G a p-group:* Given a p-group G, assume $|G| = p^r$. In view of Proposition 4.1 it suffices to prove Theorem A(a) for $L = SU(n)$. Given $f : BG \to BL_0$, where L_0 is a compact Lie group, embed $i : L_0 \subset SU(n) = L$. Obviously, $\pi_1(SU(n)/L_0)$ is finite. If $Bi \circ f \sim B\varphi$, then f is homotopic to $B\varphi_0$ for some homomorphism $\varphi_0 : G \to L_0$.

Suppose by induction that Theorem A(a) holds for G of order $\leq p^{r-1}$ and all compact Lie groups L. (The case $r = 1$ is Proposition 2.1.) Given a map $f : BG \to BSU(n)$, where $|G| = p^r$, let $V \cong \mathbf{Z}/p$ be a subgroup of the center of G, with $i : V \subset ZG$ the inclusion. If the composition $BV \xrightarrow{Bi} BG \xrightarrow{f} BSU(n)$ is null homotopic, then $f \sim f_1 \circ B\rho$ for some $\rho : G \to G/V$ and $f_1 : BG/V \to BSU(n)$ (as observed in the proof of Lemma 2.4).

By induction there is a φ_1 such that $f_1 \sim B(\varphi_1)$ and $f \sim B(\varphi_1 \circ \rho)$. Thus one may assume that $f \circ Bi \not\sim *$. By the induction hypothesis $f \circ Bi \sim B\varphi$ for some non-constant homomorphism $\varphi : V \to SU(n)$. Denote $\varphi V = C \subset SU(n)$, with $\varphi V \cong V$. Let $L_0 = Z_{SU(n)}\varphi$, with $j : L_0 \subset SU(n)$ the inclusion. By Proposition 3.4, $f : BG \to BSU(n)$ has a unique (up to homotopy) lifting $f_0 : BG \to BL_0$, with $f_0|BV = B\varphi_0$ for some $\varphi_0 : G \to L_0$.

The composition $BV \to BG \xrightarrow{f_0} BL_0 \xrightarrow{B\rho_0} BL_0/C$ is null homotopic, hence $B\rho_0 \circ f_0$ factors uniquely (up to homotopy) through BG/V: that is, one obtains a commutative homotopy pullback diagram:

$$\begin{array}{ccc} BG & \xrightarrow{f_0} & BL_0 \\ {\scriptstyle B\rho}\downarrow & & \downarrow{\scriptstyle B\rho_0} \\ BG/V & \xrightarrow{\hat{f}} & BL_0/C \end{array}$$

By induction, $\hat{f} \sim B\hat{\varphi}$ for some $\hat{\varphi} : G/V \to L_0/C$. Form the pullback diagram of groups

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\hat{\varphi}_1} & L_0 \\ {\scriptstyle \hat{\rho}}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ G/V & \xrightarrow{\hat{\varphi}} & L_0/C \end{array}$$

where $\hat{G} \cong G$. This induces a pullback diagram of classifying spaces:

$$\begin{array}{ccc} B\hat{G} & \xrightarrow{B\hat{\varphi}_1} & BL_0 \\ {\scriptstyle B\hat{\rho}}\downarrow & & \downarrow{\scriptstyle B\rho_0} \\ BG/V & \xrightarrow{B\hat{\varphi} \sim \hat{f}} & BL_0/C \end{array}$$

Obviously $f_0 : BG \to BL_0$ factors as $f_0 \sim B\hat{\varphi} \circ f_1$ for some $f_1 : BG \to B\hat{G}$. But $G, \hat{G}$ are finite groups and $f_1 \sim B\hat{\varphi}_1$, thus $f_0 \sim B(\hat{\varphi} \circ \varphi_1)$ and $f \sim B(j \circ \hat{\varphi} \circ \varphi_1)$. ∎

5. The Case of Finite Nilpotent Groups

In view of Theorem 1, if G is a finite nilpotent group and L is a compact connected Lie group, the set $[BG, BL]$ may be easily analyzed:

Let $G = \prod_{p \in \mathbb{P}_1} G_p$, where $\mathbb{P}_1$ is a finite set of primes and $|G_p| = p^{N(p)}$. As $\vee BG_p \to BG$ is a homology isomorphism one has

Lemma 5.1: *If L is connected, then $map_*(BG, BL) \to \prod_p map_*(BG_p, BL)$ is a homotopy equivalence; in particular, $[BG, BL] \cong \prod_p [BG_p, BL]$. Thus $[BG, BL] \cong \prod_p [Hom(G_p, L)/conj]$.*

Thus, the question whether a map $f : BG \to BL$ is of the form $f \sim B\varphi$ for some $\varphi : G \to L$ is equivalent to the following questions:

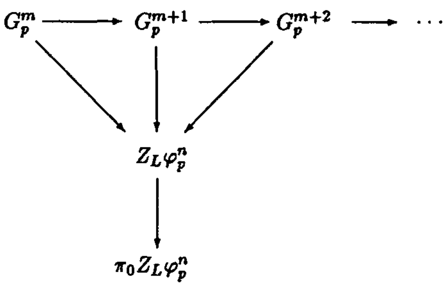
5.2 Given homomorphisms $\varphi_p : G_p \to L$ for each $p \in \mathbb{P}_1$, can one conjugate the φ_p's so that after conjugation $\varphi_p^{x_p} G_p \subset \bigcap_{q \neq p} Z_L \varphi_q^{x_q}$ $(\varphi_p^{x_p} = x_p \varphi_p x_p^{-1})$?

5.3 For the case $\mathbb{P}_1 = \{p, q\}$ the above could be formulated as follows: Given $\varphi_p : G_p \to L$ and $\varphi_q : G_q \to L$, does $Z_L \varphi_p$ contains a conjugate of $\varphi_q G_q$?

The following examples easily follow:

PROPOSITION 5.4: (a) *If G is a cyclic group, then any map $f : BG \to BL$ satisfies $f \sim B\varphi$, where the homomorphism $\varphi : G \to L$ factors through a maximal torus T_L.*
(b) *If G is abelian, then any map $f : BG \to BU(n)$ satisfies $f \sim B\varphi$, for some φ such that $\varphi G \subset T_{U(n)}$.*
(c) *If $G^n \subset G^{n+1} \subset \cdots$ is a sequence of finite abelian groups such that $\varinjlim G^n$ is divisible, then any compatible family of maps $f_n : BG_n \to BL$ factoring through $\varinjlim BG_n$ satisfies $f_n \sim B\varphi_n$ for some φ_n with $\varphi_n G_n \subset T_L$.* (d) *Given $f : BG \to BL$ there an embedding $i : L \subset U(n)$, for some n, such that $Bi \circ f \sim B\varphi$.*

Proof: (a) If G is cyclic, then $G_p \cong \mathbb{Z}/p^r$ and $\varphi_p(G_p) \subset T_L$, a maximal torus of L. After conjugation, one may assume the $\varphi_p(G_p)$ are all contained in the same maximal torus, and apply the principal of 5.2.
(b) If $L = U(n)$, any finite abelian subgroup is contained in a maximal torus. Thus again one may assume $\varphi_p G_p \subset T_L$ for the same torus T_L and all $p \in P_1$.
(c) Here for each p one has a sequence $B\varphi_p^n : BG_p^n \to BL$, with $B\varphi_p^{n+1}|BG_p^n \sim B\varphi_p^n$. Thus after conjugating φ_p^{n+1}, one may assume by induction that $\varphi_p^{n+1}|G_p^n = \varphi_p^n$. The hypothesis implies that $G_p^n \subset p^r G_p^{m(n,r)}$. Now $\varphi_p^m(G_p^m) \subset Z_L\varphi_p^n$ for all $n \leq m$, and $Z\varphi_p^m \subset Z_L\varphi_p^n$ for $n \geq m$. So the homomorphisms



yield a map $\lim_{\to} G_p^m \to \pi_0 Z_L\varphi_p^n$. Since the group $\pi_0 Z_L\varphi_p^n$ is finite, this morphism, and all $G_p^m \to \pi_0 Z_L\varphi_p^n$, are trivial. Thus $\varphi G_p^m \subset (Z_L\varphi_p^n)_0$ for all $m \geq n$.

Now $(Z_L\varphi_p^n)_0 \supset (Z_L\varphi_p^{n+1})_0 \supset \cdots$ is a descending sequence of compact connected Lie groups, so it must stabilize — i.e., for some m, $Z_L\varphi_p^m = Z_L\varphi_p^{m+1} = \cdots = L_0$, and all $\varphi_p^m G_p^m$ are contained in the center of a compact connected Lie group, hence in a maximal torus. Now proceed as in (a) and (b).
(d) It suffices to prove this for $L = U(n)$. If $|\mathbb{P}_1| = m$ one can easily see that the composite $BG \to BU(n) \to BU(n,m)$ is homotopic to a map of the form $B\varphi$,

where φ is given by

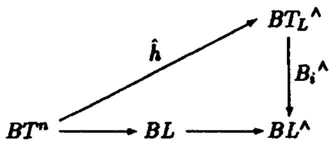
$$\varphi_{p_1} \times \cdots \times \varphi_{p_m} : G_{p_1} \times \cdots \times G_{p_m} \to U(n) \times \cdots \times U(n) \hookrightarrow U(m). \qquad \blacksquare$$

Example 5.5: The smallest example of an abelian group G with a map $f : BG \to BL$ for which $f \not\sim B\varphi$ is the following:

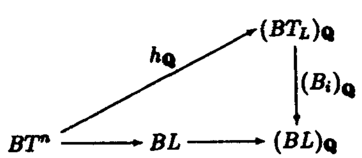
Let $L = SO(3)$ and $G = \mathbb{Z}/2 \times \mathbb{Z}/6$. Let $\varphi_2 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to SO(3)$ be generated by two 180° rotations around perpendicular axes. Let $\varphi_3 : \mathbb{Z}/3 \to SO(3)$ be the (unique up to conjugacy) embedding. Since $Z_L\varphi_2 = \mathrm{im}\varphi_2$, no conjugate of $\mathrm{im}\varphi_3$ is contained in $Z_L\varphi_2$ and the map $f : BG \to BSO(3)$ satisfying $f|BG_{p_i} = B\varphi_{p_i}$ $(p_i = 2, 3)$ is not homotopic to $B\varphi$ for any $\varphi : G \to L$. $\blacksquare$

6. The Case G is a Torus

Let $G = T^n$ be the n-torus. Assume given a map $f : BG \to BL$, where L is a compact and (without any loss of generality) connected Lie group. Let $V_m^n = (\mathbb{Z}/m!)^n \hookrightarrow T^n$. By Proposition 5.4(c), $BV_m^n \xrightarrow{B\sigma} BT^n \xrightarrow{f} BL$ lifts as $BV_m^n \xrightarrow{f_0} BT_L \xrightarrow{Bi} BL$ in a coherent way. Hence $\lim_{\to} BV_m^n = BV_\infty^n \to BT^n$ also lifts to BT_L. Since $[BT^n, K^\wedge] \xrightarrow{\cong} [BV_\infty^n, K^\wedge]$ for any profinite complete space $K^\wedge$, one obtains a lifting



By [AM], one always has a lifting



(where $(-)_\mathbb{Q}$ is the rationalization).

To obtain a map $BT^n \to BT_L$ out of $\hat{h}$ and $(h)_\mathbb{Q}$ using the arithmetic square, one needs the homotopy equivalence of the following composites:

$$\begin{array}{ccccc} BT^n & \xrightarrow{\hat{h}} & B\hat{T}_L & \xrightarrow{\hat{V}_1} & (B\hat{T}_L)_\mathbb{Q} \\ BT^n & \xrightarrow{h_\mathbb{Q}} & (BT_L)_\mathbb{Q} & \xrightarrow{\hat{V}_2} & (B\hat{T}_L)_\mathbb{Q} \end{array}$$

Now comparing these two maps with $(B\hat{T}_L)_{\mathbb{Q}} \to (B\hat{L})_{\mathbb{Q}}$ one obtains an inequality. Moreover, as BT_L, BL are rational H-spaces, $(B\hat{T}_L)_{\mathbb{Q}}$ and $(B\hat{L})_{\mathbb{Q}}$ are products of Eilenberg–MacLane spaces with (finitely many) nonvanishing homotopy groups, concentrated in even dimensions — all being $\mathbb{Z}^\wedge \otimes \mathbb{Q}$-vector spaces. Thus the problem of $\hat{v}_1 \circ \hat{h} \sim \hat{v}_2 \circ h_{\mathbb{Q}}$ is just a matter of equality of morphisms of $H^*(-, \mathbb{Z}^\wedge \otimes \mathbb{Q})$:

$$H^*(BL, \mathbb{Z}^\wedge \otimes \mathbb{Q}) \xrightarrow{Bi^*} H^*(BT_L, \mathbb{Z}^\wedge \otimes \mathbb{Q}) \underset{(v_2 \circ h_{\mathbb{Q}})^*}{\overset{(\hat{v}_1 \circ \hat{h})^*}{\rightrightarrows}} H^*(BT^n, \mathbb{Z}^\wedge \otimes \mathbb{Q}),$$

where $(\hat{v}_2 \circ h_{\mathbb{Q}})^* Bi^* = (\hat{v}_1 \circ \hat{h})^* Bi^*$

Again, by [AM, Theorem 1.7], $(\hat{v}_1 \circ \hat{h})^*$ and $(\hat{v}_2 \circ h_{\mathbb{Q}})^*$ differ by an element w in the Weyl group of BT_L: $(\hat{v}_1 \circ \hat{h})^* = (\hat{v}_2 \circ (h_{\mathbb{Q}})^*) \circ w^*, w : BT_L \to BT_L$. Thus, changing, say, $h_{\mathbb{Q}}$ to $w \circ h_{\mathbb{Q}}$, we may assume that the equality $[\hat{v}_1 \circ \hat{h}] = [\hat{v}_2 \circ h_{\mathbb{Q}}]$ holds. One thus obtains a map $h : BT^n \to BT_L$ so that $\hat{B}i \circ \hat{h} \sim \hat{f}$ and $Bi_{\mathbb{Q}} \circ h_{\mathbb{Q}} \sim f_{\mathbb{Q}}$. The obstruction to $Bi \circ h \sim f$ lies in $H^{odd}(BT^n, \pi_*(B\hat{L})) = 0$ and thus, combining with Proposition 2.1, one obtains:

PROPOSITION 6.1 (Theorem A(a) for $G = T^n$): *Any map $f : BT^n \to BL$ lifts to a map $h : BT^n \to BT_L$; hence $f \sim B\varphi$ for some $\varphi : T^n \to L$.*

To prove Theorem A(b) for G a torus, one notices the following: Let $\overline{V}_r^n = (\mathbb{Z}/p^r)^n \subset T^n$. Given $\varphi_1, \varphi_2 : T^n \to L$ with $B\varphi_1 \sim B\varphi_2$, by Theorem A(b) for G a p-group we have $\varphi_1 | \overline{V}_r^n \sim \varphi_2 | \overline{V}_r^n$.

Now $Z_L(\varphi_1 V_r) \supset Z_L(\varphi_1 V_{r+1}) \supset \cdots$ must stabilize — that is, $Z_L(\varphi_1 V_{r_0}) = Z_L(\varphi_1 v_{r_0+1}) = \cdots = Z_L \varphi_1 T^n$. Suppose $\varphi_2 | V_{r_0}^n = x_{r_0}(\varphi | V_{r_0}^n) x_{r_0}^{-1}$; replacing φ_2 by $\varphi_2^1 = x_{r_0} \varphi_1 x_{r_0}^{-1}$, if necessary, one may assume that $\varphi_1 | V_{r_0}^n = \varphi_2 | V_{r_0}^n$. Now for $r > r_0$, $\varphi_2 | V_r = x_r \varphi_1 | V_r^n x_r^{-1}$. However, $\varphi_2 | V_{r_0}^n = \varphi_1 | V_{r_0}^n$ implies $x_r \in Z_L(\varphi_1 | V_{r_0}) = Z_L(\varphi_1 | V_r^n)$; hence $\varphi_2 | V_r^n = \varphi_1 | V_r^n$ for all $r \geq r_0$, so $\varphi_2 | V_\infty^n = \varphi_1 | V_\infty^n$. Since V_∞^n is dense in T^n, this implies $\varphi_2 = \varphi_1$.

7. Extending the Torus

Proof of Theorem B: (a) Let $V_r = (\mathbb{Z}/p^r)^n \subset T^n = T$. Given $\varphi : T \to L$, denote by φ_r its restriction to V_r. Then there is an r_0 such that $Z_L \varphi_r = Z_L \varphi$ for all $r \geq r_0$. By Proposition 3.5(a)

$$\mathrm{map}(BV_r, B((Z_L\varphi)_p^\wedge)_{B\hat{\varphi}_r} \xrightarrow{\cong} \mathrm{map}(BV_r, B(L_p^\wedge))_{B\hat{\varphi}_r}$$

and it also follows from Proposition 3.5(b) that $\mathrm{map}_*(BV_r, B(Z_L\varphi_p^\wedge))_{B\hat\varphi_r} \cong *$. As $\mathrm{map}(BT, X_p^\wedge)_f \to \lim_{\leftarrow} \mathrm{map}(BV_r, X_p^\wedge)_{f_r}$, one has

$$\begin{aligned}
&\mathrm{map}(BT, B((Z_L\varphi)_p^\wedge))_{B\hat\varphi} \xrightarrow{\cong} \mathrm{map}(BT, B(L_p^\wedge))_{B\hat\varphi}\\
&\mathrm{map}_*(BT, B((Z_L\varphi)_p^\wedge))_{B\hat\varphi} \simeq *\\
&\mathrm{map}(BT, B((Z_L\varphi)_p^\wedge))_{B\hat\varphi} \xrightarrow{\cong} B((Z_L\varphi)_p^\wedge)\,.
\end{aligned}$$

These three together yield

$$\mathrm{map}_*(BT, B(L_p^\wedge))_{B\hat\varphi} \xrightarrow{\cong} R_p(L/Z_L\varphi)\,.$$

As this holds for all p, one has

$$\mathrm{map}_*(BT, B(L^\wedge))_{B\hat\varphi} \xrightarrow{\cong} (L/Z_L\varphi)^\wedge\,.$$

To compare $\mathrm{map}_*(BT, BL)_{B\varphi}$ with $\mathrm{map}_*(BT, BL^\wedge)_{B\hat\varphi}$, where $BL^\wedge = B(L^\wedge) = (BL)^\wedge$, one uses the arithmetic square

$$\begin{array}{ccc}
BL & \longrightarrow & BL^\wedge\\
\downarrow & & \downarrow\\
BL_{\mathbb{Q}} & \longrightarrow & (BL^\wedge)_{\mathbb{Q}}
\end{array}$$

As $BL \approx_{\mathbb{Q}} \prod_{i=1}^r K(\mathbb{Z}, 2n_i)$, we have $BL^\wedge \approx_{\mathbb{Q}} \prod_{i=1}^r K(\mathbb{Z}^\wedge, 2n_i)$, and the bottom map in the above diagram is actually the map

$$\prod_{i=1}^r K(\mathbb{Q}, 2n_i) \to \prod_{i=1}^r K(\mathbb{Q}^\wedge, 2n_i),$$

where $\mathbb{Q}^\wedge = \mathbb{Z}^\wedge \otimes \mathbb{Q}$. Hence $BL \to BL^\wedge$ is a principal fibration induced by a map

$$BL^\wedge \to \prod_{i=1}^r K(\mathbb{Q}^\wedge/\mathbb{Q}, 2n_i) = \prod_{i=1}^r K(\mathbb{Z}^\wedge/\mathbb{Z}, 2n_i)$$

and one has a fibration

$$\mathrm{map}_*(BT, BL)_{B\varphi} \to \mathrm{map}_*(BT, BL^\wedge)_{B\hat\varphi} \to \mathrm{map}_*(BT, \prod_i K(\mathbb{Z}^\wedge/\mathbb{Z}, 2n_i))\,.$$

The latter is equivalent to $\prod_{j=1}^s K(\mathbb{Z}^\wedge/\mathbb{Z}, 2m_j)$, $m_j > 0$ and Theorem B(a) follows.

For (b), note that if X is a rational H-space with $\pi_{odd}X = 0$, the homotopy fiber of $\mathrm{map}(BT, X)_f \to \mathrm{map}(BT, X^\wedge)_{\hat{f}}$ is a product of Eilenberg–MacLane spaces whose homotopy groups are finite dimensional $\mathbb{Z}^\wedge/\mathbb{Z}$ modules — hence rational H-spaces. As

$$\mathrm{map}(BT, B((Z_L\varphi)^\wedge))_{B\hat{\varphi}_0^1} \xrightarrow{\cong} \mathrm{map}(BT, BL^\wedge)_{B\hat{\varphi}_0},$$

part (b) of the Theorem follows. ∎

THEOREM C: *Let* $0 \to T \xrightarrow{\sigma} G \xrightarrow{\tau} W_G \to 1$ *be a finite extension of a torus* $T = T^n$, *let* L *be a compact connected Lie group, and let* $f : BG \to BL$ *satisfy* $f|BT \sim B\varphi$ *for some homomorphism* $\varphi : T \to L$. *Then:*

(a) *There exists a homotopy commutative diagram*

$$\begin{array}{ccc} BT & \xrightarrow{B\varphi_0} & BZ_L\varphi \\ {\scriptstyle B\sigma}\downarrow & & \downarrow{\scriptstyle B\sigma'} \\ BG & \xrightarrow{f'} & BN_L\varphi \\ {\scriptstyle B\tau}\downarrow & & \downarrow{\scriptstyle B\tau'} \\ BW_G & \xrightarrow{B\alpha} & BW_L\varphi \end{array}$$

where $\alpha : W_G \to W_L\varphi$ *is a homomorphism and* $\varphi_0 : T \to Z_L\varphi$ *is induced by* φ, *and* f' *covers* f *up to homotopy.*

(b) *If* $Z_L\varphi$ *is a torus, then* $f' \sim B\varphi'$ *for some homomorphism* $\varphi' : G \to N_L\varphi$. *In particular, if* L *is simple, any map* $f : BL \to BL$ *induces an endomorphism* $\varphi_1 : NT_L \to NT_L$ *such that* $B\varphi_1$ *is compatible with* f.

Proof: (a) $\varphi : T \to L$ factors through $\varphi_0 : T \to Z_L\varphi$. Thus $BT \to BG \xrightarrow{f} BL$ factors through $BZ_L\varphi$.

Given $\omega \in W_G$, let $a_\omega : G \to G$ be the inner automorphism induced by any representative of ω with $a_\omega T \subset T$. Then $Ba_\omega \sim 1$ and thus $B\varphi \circ Ba_\omega \sim B\varphi$ and, by Theorem A(b), φ and φa_ω are conjugates, so there is an $x \in L$ with $\varphi a_\omega = x\varphi x^{-1}$. But as $\mathrm{im}\varphi = \mathrm{im}\varphi a_\omega$, $x \in N_L\varphi$, the class of x in $N_L\varphi/Z_L\varphi = W_L\varphi$ is uniquely determined. One can easily see that the assignment $\omega \to [x] \in W_L\varphi$ induces a homomorphism $\alpha : W_G \to W_L\varphi$. Moreover, $BZ_L\varphi$ admits a $W_L\varphi$-action with

$$BZ_L\varphi \times_{W_L\varphi} EW_L\varphi = BN_L\varphi$$

and the map $B\varphi_0 : BT \to BZ_L\varphi$ is an α-map. Equivalently, if one considers $BZ_L\varphi$ as a W_G-space under the action induced by α, then $B\varphi_0$ is W_G-equivariant.

Now

$$\mathrm{map}(BG, BL)_{[B\varphi]} \cong \mathrm{map}_{W_G}(BT, BL)_{(B\varphi)}$$
$$\mathrm{map}_{W_G}[(BT, BZ_L\varphi)_{(B\varphi_0)}] \cong \mathrm{map}(BG, B\hat{N})_{[B\varphi_0]}$$

where $\hat{N}$ is the pullback

$$\begin{array}{ccc} \hat{N} & \longrightarrow & N_L\varphi \\ \downarrow & & \downarrow \\ W_G & \xrightarrow{\alpha} & W_L\varphi \end{array}$$

By Theorem B(b), the fiber of $\mathrm{map}(BT, BZ_L\varphi)_{B\varphi_0} \to \mathrm{map}(BT, BL)_{B\varphi}$ has rational homotopy groups hence any W_G-map $EW_G \to \mathrm{map}(BT, BL)_{B\varphi}$ lifts uniquely to a W_G-map $EW_G \to \mathrm{map}(BT, BZ_L\varphi)_{B\varphi_0}$, and thus $BG \to BL$ lifts to $B\hat{N}$, hence to $BN_L\varphi$.

For (b): if L is simple and $f : BL \to BL$ has $H^*(f, Q) \neq 0$, then $H^*(f, Q)$ is an isomorphism. Let $f|T^n = B\varphi$. By (a) one has

$$\begin{array}{ccc} BT & \longrightarrow & BZ_L\varphi \\ \downarrow & & \downarrow \\ BN_T & \xrightarrow{fN} & BN_L\varphi \\ \downarrow & & \downarrow \\ BW_L & \longrightarrow & BW_L\varphi \end{array}$$

But $H^*(f, Q)$ is an isomorphism, hence $\varphi T \subset L$ is again a maximal torus T_L and $Z_L = T_L$ and, by Proposition 2.3, $f_N \sim B\varphi_1$. ∎

8. Editor's Postscript

The proposition below is easily obtained by combining the results of this paper with facts from Lie theory,

PROPOSITION: *Let $f : BL \to BH$ be a homotopy equivalence of compact, connected, semi-simple Lie groups. Then there is an isomorphism of groups, $L \cong H$.*

Proof: Let T_L be a maximal torus of L. Restriction of f to BT_L and an application of Theorem A yields a homomorphism $\varphi : T_L \to H$ such that the restriction of f to BT_L is homotopic to $B\varphi$. The subgroup $\mathrm{im}\varphi$ is compact, connected and abelian, hence is a torus T. Let $g : BH \to BL$ be a homotopy inverse to f. A similar application of Theorem A yields a homomorphism $\psi : T \to L$ such that g restricted to BT is homotopic to $B\psi$. Since $g \circ f$ is homotopic to the identity, Theorem A(b) yields an element $x \in L$ such that

$$\psi \circ \varphi_0 = xix^{-1}$$

where $\varphi_0 : T_L \to T$ is the factorization of φ through T and $i : T_L \to L$ is the inclusion of the maximal torus. Thus we can replace g by $Ba_{x^{-1}} \circ g$ to obtain the following diagram:

$$\begin{array}{ccccc} BT_L & \xrightarrow{B\varphi_0} & BT & \xrightarrow{B\psi_0} & BT_L \\ \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle i} \\ BL & \xrightarrow{f} & BH & \xrightarrow{g} & BL \end{array}$$

in which the squares commute up to homotopy, g and f are an inverse pair up to homotopy (as $Ba_{x^{-1}} \sim \mathrm{id}$) and φ_0, ψ_0 are an inverse pair of homomorphisms. In addition, we have factored ψ_0 through a maximal torus, which perforce is T_L. Consequently, T is a maximal torus, now denoted T_H.

The construction in the proof of Theorem C gives a homomorphism $\alpha : W_L \to W_H\varphi$. Since $\mathrm{im}\varphi_0$ is a maximal torus, we have $Z_H\,\varphi_0 = T_H$ and $W_H\,\varphi = W_H$.

Hence we have the following commutative diagram:

$$\begin{array}{ccc} W_L \times T_L & \longrightarrow & T_L \\ {\scriptstyle \alpha \times \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ W_H \times T_H & \longrightarrow & T_H \end{array}$$

Since φ_0 is surjective and the action of W_H on T_H is effective, there is a unique homomorphism α in the above diagram. The same considerations applied to ψ_0 give a unique homomorphism $\beta : W_H \to W_L$ such that $\psi_0\varphi_0$ is compatible with $\beta\alpha$. Since ψ_0 and φ_0 form an inverse pair, so must α and β. Now Theorem C provides a homomorphism $\sigma : N_L \to N_H$ such that $B\sigma$ and; *a fortiori*, σ is a homotopy equivalence.

To see that the homomorphism σ is an isomorphism, we note that both N_L and N_H are compact, orientable closed manifolds and $\deg \sigma = \pm 1$ on each component, so σ is surjective. Hence σ may be regarded as a covering projection. Since $\ker \sigma$ is discrete and σ is an equivalence, in fact $\ker \sigma$ must consist of the identity alone. Hence σ is an isomorphism. The main result in [CWW] asserts that, under the hypotheses of the proposition, an isomorphism of normalizers N_L with N_H implies an isomorphism of L with H. ∎

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